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Fracton mediated superconductivity in the “net fractal”systems. Preliminaries and an overview of other fracton based models

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Assuming that the force constants scale as $\sigma(\lambda x) = \lambda^{-\alpha}\sigma(x)$, we construct the model of elastic (linear) excitations on a fractal (fractons). We show that the fractons on a specific class of fractals, “net fractals”, can be assumed to be log-scale phonons. Further, we discuss the model of fracton mediated superconductivity in the “net fractal” systems). We show that with the use of logarithmic coordinates, the fracton mediated superconductivity can be described within a model which is reminiscent of the conventional BCS formulation.

Key words: *fractals; fractons; superconductivity*

1. Introduction

The concept of fractal geometry has proven useful in describing structures and processes in nanoscale many-body systems [1]. The hallmark of a fractality is a hierarchical organization of its elements, described by discrete scaling laws, which makes the fractal, regardless the magnification or contraction scale, look the same. This property of fractals is called self-similarity, self-affinity or self-replicability. Although physical systems modellled by fractals are non-translation invariant, it is well known fact that self-similar fractals as well as physical quantities on fractal systems show log -periodicities [1]. This opens a possibility to describe the symmetries of self-similar fractals (with the use of logarithmic scale), in a way that is reminiscent of conventional formalism developed for crystalline systems [2]. Motivated by this fact, we present study of superconductivity, when the electron pairing is mediated by fractal excitations (fractons) [3].

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A self-similar symmetry of a fractal is a transformation that leaves the system invariant, in the sense that, taken as a whole it looks the same after transformation as it did before, although individual points of the pattern may be moved by the transformation. We say that $\mathbf{K} \subset \mathbf{R}^n$ satisfies the scaling law $\mathbf{S}$, or is a self-similar fractal, if $\mathbf{S}:\mathbf{K} = \mathbf{K}$. Let us limit our considerations to fractals in which the self-similarity can be realized only via linear maps, i.e., transformations which point $\mathbf{r} = (x_1, x_2, x_3) \in \mathrm{K} \subset \mathbf{R}^3$ transform into point $\mathbf{r}' = (x_1' x_2', x_3')$ according to the formula $x_i' = \mathrm{S}_{i1}x_1 + \mathrm{S}_{i2}x_2 + \mathrm{S}_{i3}x_3$; where $i = 1, 2, 3$. The vector form of the linear self-similar transformation can be written as $\mathbf{r}' = \mathbf{S}:\mathbf{r}$, where $\mathbf{S}$ is the matrix of the linear self-similar transformation. If we orient coordinate axes along the eigenvectors of matrix $\mathbf{S}$ (i.e., $\mathbf{x} = (x_1, x_2, x_3) \rightarrow (\varepsilon, \eta, \rho)$), the linear self-similar mapping reduces to the transformation $\mathbf{S}$: $(\varepsilon, \eta, \rho) \rightarrow (\lambda_1\varepsilon, \lambda_2\eta, \lambda_3\rho)$. In the case of infinite-size fractals also the inverse $\mathbf{S}^{-1}$ mapping fulfils the self-similarity conditions $\mathbf{S}^{-1}:\mathbf{K} = \mathbf{K}$ and for any $\mathbf{x} \in \mathbf{K}$, we have

$$S^{-1}:x = S_1^{-1}\cdot S_2^{-1}\cdot S_3^{-1}:x = (\lambda_1^{-1}\varepsilon, \lambda_2^{-1}\eta, \lambda_3^{-1}\rho) \qquad (1)$$

Consider a more general transformation of the type $\mathrm{S}^{(m,n,l)} = (\mathrm{S}_1)^n\cdot(\mathrm{S}_2)^m\cdot(S_3)^l$, where $(S_i)^n$ denotes n-tuple superposition of transformation S_i, and define a class of infinite "net fractals" G_{nf}, for which the relation $\mathrm{S}^{(m,n,l)}$: $G_{nf} = G_{nf}$ is valid. Action of $S^{(m,n,l)}$ transforms any point $x \in \mathbf{R}^3$ according to the formula $S^{(m,n,l)}:x = (\lambda_1^n\ \varepsilon, \lambda_1^m\eta, \lambda_3^l\rho)$, where m, n, l are arbitrary (negative or positive) integers. In view of this relation, we have that $S^{(m,n,l)}:G_{nf} \subset G_{nf}$, i.e., $S^{(m,n,l)}$ are the injective scaling mappings. For any linear S_1 and $F_1 \subset \mathbf{R}$ by definition we have $S_1:F_1 = F_1$ and for any $x_0 \in F_1$ we have S_1: $x_0 = \lambda_1 x_0$, consequently $(\mathrm{S}_1)^m$: $x_0 = \lambda_1^m: x_0$. Using the logarithmic scale, we have $\log(x_m/x_0) = m\ln\lambda_1$ ($m = \pm1, \pm2, \ldots$). This is nothing but a 1D crystal lattice with the lattice spacing given by $a_1 = \ln\lambda_1$. Using the multi-logarithmic scale, we can see that the family of mappings $\mathrm{S}^{(m,n,l)}$ is isomorphic with a 3D crystal lattice. This means that the isomorphism $\mathrm{S}^{(m,n,l)} \leftrightarrow (ma_1, na_2, la_3)$ holds. The very same refers to the placement of its characteristic building blocks.

To show the crystal structure of self-similar fractal in the log scale, let us consider the triadic Cantor set (CS). The Cantor set is created by repeatedly deleting the open middle thirds of the interval [0, 1]. The construction of the CS starts by deleting the open middle third leaving the two segments $[0, 1/3] \cup [2/3, 1]$. In the next step, the open middle third of each remaining segments is left behind. The process is continued *ad infinitum*. Let us now picture this procedure in the logarithmic, $\log_3$ scale coordinates. The initial interval [0, 1] in the $\log_3$ scale is mapped onto the half line $[-\infty, 0]$. We can illustrate this mapping in the following way: the interval [0, 1] can be presented as an infinite sum of disjoint subsets $\mathrm{T}_n = [3^{-n-1}, 3^{-n}]$. In the $\log_3$ scale, each T_n is transformed into the interval $\mathrm{t}_n = [-n-1, -n]$ being the unit cell of the half-infinite, log scale crystal. The set obtained after the first step of the Cantor procedure (pictured in the $\log_3$ scale) is the union of two intervals $[-\infty, -1] \cup [\log_3 2-1, 0]$. After the second step, the log picture of Cantor procedure is given by the union of intervals $[-\infty, -2] \cup [\log_3 2, -2] \cup [\log_3 2-1, \log_3 7-2] \cup [\log_3 8-2, 0]$, etc. We can see that at every step k

the number of segments is doubled, and the picture of points belonging to the interval $[-n-1, -n]$ is identical with the picture of those points of the CS belonging to the preceding unit interval $[-n, -n+1]$ at the preceding stage of Cantor construction. The only exception is the appearance of subset $CS_{[-1,0]}$, i.e., of the points of the CS set that, in the $\log_3$ scale, fall into the $[-1, 0]$ interval. At every step of the Cantor procedure, there arise essential changes in the appearance of the $CS_{[-1,0]}$ subset which always gains a novel, more complicated structure. The $\log_3$ picture of the first few steps of the $CS_{\log}$ construction is presented in Fig. 1.

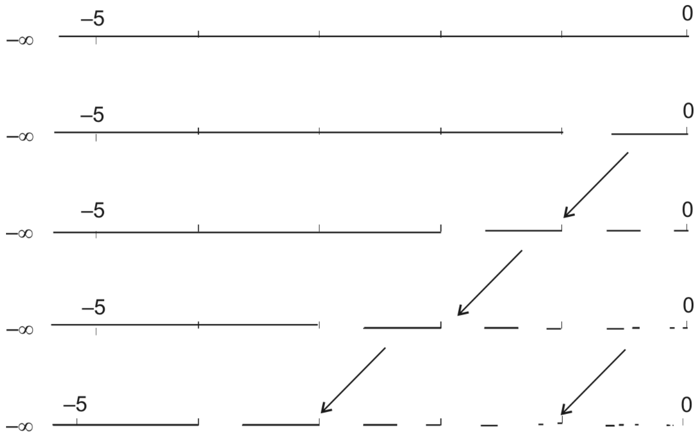

Fig. 1. First few steps in constructing the Cantor set in the $\log_3$ scale

We can summarize the log scale construction as follows. After every step of the Cantor construction there arises new structure of the CS $_{\log}{}^{[01]}$ subset, while the remaining part of the picture is identical with that obtained after preceding step of the Cantor procedure, although moved to the left by a unit segment. This means that each unit interval $[-n-1, -n]$ undergoes the same reduction scheme (but with some delay with respect to the number of steps). Consequently, when the Cantor procedure is continued ad infinitum each unit segment $[-n-1, -n]$ becomes identical. As an effect of that the Cantor set (in the log scale) is mapped onto semi-infinite 1D crystal lattice. Evidently, if we consider the infinite-size CS_∞ in the $\log_3$ scale picture, it covers the unlimited 1D crystal lattice. We should point out here, that the "unit cell" of this lattice has a complex, Cantor like, structure.

2. Fractons

The purpose of this paper is to study the vibrations in a system deformable over a fractal subset. Most theoretical studies of the vibrations of a fractal limit considerations to a universal level without referring to the specific physical model. In our study, we focus our considerations on a specific model which, we believe, describes the behaviour of some real systems. Consider a "net fractal" cluster as defined above , consisting of N atoms with unit mass and linear springs connecting nearest-neighbour sites. The equations of motion of the atoms are [4]

$$\ddot{u}_n(t) + \sum_m k_{n,m} u_n(t) = 0 \tag{2}$$

where the sum goes over all nearest neighbours sites of the fractal site n. When trying to work with Eq. (2), one meets two problems, first is that elastic constants $k_{m,n}$ and the mass distribution depend on the coordinates and the second is associated with the ambiguities in the definitions of the local displacements u_n. The local strain e ($e \propto \nabla u(r,t)$) and local displacements on the fractal system can be defined in two ways. One refers to the internal geometry and microscopic interactions, while the other defines the strain directly in terms of the effect of deformations on the (suitably averaged) mass distribution.

Since we are interested in the study of fractal acoustics, we should use the latter definition, which is directly relevant to the experiment. As was pointed by Alexander [4], in this case the vibrational displacements are the vectors in the embedding space and are not restricted by the internal geometry of the fractal. Let us discuss now the non-homogeneities of mass and force constants. Due to rapid fluctuations on short length scales, the strains and density can be defined only as the scale dependent local averages [4]. We can therefore assume that fractal of the size r has, on the average, a mass $m(r) = m_0(r/a)^d$, where d is the fractal mass dimension. It is natural to assume that the self-similarity of the fractal is reflected also in the dilation symmetry. Assuming that ω is the eigenfrequency of the fracton oscillations, we can find that the force constants k_i scale as $k_i = m_k\omega_k^2 \propto (r/a)^d\omega_k^2$. In view of the latter relation, from here on, we assume that the forces which tend to restore the equilibrium positions of species, are linear (with respect to the coordinates of the excited fractal system). However, contrary to the conventional solid, the elastic constants are not homogeneous and depend on the coordinates. Let us assume that elastic forces follow the common power law scaling with the separation [4]. As we have shown above, when presented in the logarithmic coordinates, the mass density of such a fractal becomes uniform; the same refers to the elastic constants. Suppose the fractal is perturbed locally (e.g., in the vicinity of the equilibrium position x_0, with the energy ε_0) and consider the amplitude of this excitation. In a real space, the amplitude of a local fluctuation has the form $u_n = \left|x_n^0 - x_n\right|$, while in the log coordinates we have $\zeta_n = \left|\xi_n^0 - \xi_n\right|$, where $\zeta_n = \ln x_n$. Consider first a somewhat unrealistic case when there are no broken bonds in the log scale picture. In this case (in the log scale) we have a homogeneous system with a uniform mass and elastic constant distribution. Under conditions above, application of the continuous medium approximation is justified. Thus, when perturbed, the log coordinates ζ_n and the local displacement $\zeta_n(x, t)$ should satisfy the classical wave-equation $\nabla^2\zeta - (1/c^2)(\partial^2\zeta/\partial t^2) = 0$ with the plane wave solution $\zeta_i = \zeta_i^0 \exp(ik_i\zeta_i - i\omega t)$

$= \zeta_i^0(x_i)^{ik}_i e^{i\omega t}$, when the relation $\zeta_n = \ln x_n$ is taken into account. As we can see from above, the fracton appears to be the log scale phonon. When transformed to physical space, the log scale phonon solution displays power law scaling with purely imaginary scaling exponent. The extensive discussion of the systems with complex scaling factors was given by Sornette [5], who proved that this type of scaling results in the log –periodic oscillations of physical quantities.

3. Fracton mediated superconductivity

The conjecture that superconductivity comes about because of the fractal structure of underlying medium was raised firstly by Buettner and Blumen [6] in discussion of the high-temperature superconductivity (HTC). In the HTC of copper oxides, the onset of superconductivity is closely related to the oxygen deficiency. It was postulated that the oxygen vacancies located mainly within the CuO_2 planes of the YBCO system form fractal structures. Since the fracton vibrational frequency cutoff ω_{FD} is much greater than the Debye frequency of crystalline systems, there arose a conjecture that conduction electron scattering of fractons can be responsible for the high critical temperature [7]. In the following, using the results obtained above, we will show that in the log coordinates, the Hamiltonian of the fracton based superconductivity receives the conventional BCS form. The fact that fractons can be expressed as the log -phonons suggests that they (or at least some of them) are bosons. This supports the idea of fracton pairing in superconductors [6–10]. Since in the log scale the "net fractal" system becomes uniform (uniform density and uniform elastic forces), in description of elastic energy we can limit ourselves to the harmonic approximation. Consequently, the Hamiltonian of the elastic system can be expressed with the help of the fracton creation ϕ^+ and destruction ϕ operators as:

$$H_{fr-\lg} = \sum_k \hbar\omega(\phi_k^+\phi_k + \frac{1}{2}) \tag{3}$$

We can write the explicit form of the fracton ϕ^+ creation and destruction ϕ operators

$$\phi^+ \propto \xi - \frac{d}{d\xi} = \ln x - x\frac{d}{dx} \tag{4}$$

The scenario presented above supports the idea of fracton mediated superconductivity raised by Rasmussen and Milovanov [3]. By the analogy to the conventional solids where the phonons can mediate electron pairing (BCS theory), they postulated that in the self-similar systems the fractons can play the role of phonons. Within our approach, in the log scale their model receives a novel form which exactly matches the BCS formulation in the log scale. The electron placed on a fractally organized system produced structural deformation which couples the other electron. Assuming linear electron-fracton coupling, the effective interaction reads

$$H = \sum_{k,k'} W_{k,k'} \phi_{k'-k} c^{+}_{k'} c_k + \sum_{k,k'} W^{*}_{k,k'} \phi^{+}_{k'-k} c^{+}_{k} c_{k'} \tag{5}$$

where C^{+} is the creation operator for an electron with the (log scale) wave vector **k**, ϕ^{+} $(k' - k)$ is the creation operator for a fracton, and $W_{k',k}$ are the elements of interaction matrix. The electron–fracton coupling leads thus to the BCS-like Hamiltonian that produces the Cooper pair formation

$$H = \sum_{\vec{k},\sigma} (\in_k - \mu) c^{+}_{k,\sigma} c_{\vec{k},\sigma} + \sum_{k,k_1} V_{kk_1} c^{+}_{k,\uparrow} c_{-k,\downarrow} c^{+}_{k_1,\uparrow} c_{-k,\downarrow} \tag{6}$$

where $c^{+}_{k,\sigma}$ is the fermion creation operator labeled by the (log scale) wave vector **k** and spin σ.

4. Discussion and summary

A characteristic feature of superconducting copper oxides is their spatial inhomogeneity of oxygen stoichiometry on mesoscopic scales. When approaching some critical oxygen concentration p_c, there arise ramified (fractal) clusters for which both static and dynamic quantities show spatial, power-law scaling [10]. The sample can be considered as the collection of fractal systems immersed in some matrix. In view of the above assumption that the fracton based mechanism can contribute to the formation of superconducting state in the HTC copper oxides is fully justified. Generally one would expect degradation of superconducting state with clustering. Also a detailed study of the fracton based superconductivity leads to the conclusion [9] that presence of fractons does not result in an automatic increase of critical temperature. Only under certain circumstances is the T_C of fractal system higher than in the bulk system. However, due to the fractality there arises a mechanism which can compensate the destructive effect of fractality on the superconducting state. Let us consider the density of elementary excitations $n(\varepsilon)$ in the collection of fractal systems, in the simplest case it behaves as $n(\varepsilon)d\varepsilon = (\varepsilon - \varepsilon_0)D_{\mathrm{eff}}/2 - 1$ [11] , where D_{eff} denotes the effective fracton (spectral dimension). It worth to note that some systems can exhibit fracton dimension much higher than the topological one, e.g., the density of states in the irradiated GaN structures, depending on the degree of irradiation that generates fractal structures varies within the $0.86 < D < 4.7$ [12]. It can be shown that the values of effective fracton dimension D_{eff} of the quasi-2D quasiparticle system (e.g., Cooper pairs within the CuO_2 planes of YBCO) can vary within the $1 < D_{\mathrm{eff}} < 4$ range. The value of D_{eff} has important implications on superconducting phase transitions. Provided that the relation $D_{\mathrm{eff}} > 3$ holds in some energy window close to the Fermi energy, this can lead to the elevated critical temperatures [11].

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